ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 10, pp. 1221–1231.

Suppressing Exogenous Disturbances in a Discrete-Time Control System as an Optimization Problem

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Received June 19, 2023
Revised July 24, 2023
Accepted August 2, 2023

Abstract—This paper proposes a novel approach to suppressing bounded exogenous disturbances in a linear discrete-time control system by a static state- or output-feedback control law. The approach is based on reducing the original problem to a nonconvex matrix optimization problem with the gain matrix as one variable. The latter problem is solved by the gradient method; its convergence is theoretically justified for several important special cases. An example is provided to demonstrate the effectiveness of the iterative procedure proposed.

Keywords: linear discrete-time system, exogenous disturbances, output feedback, state feedback, optimization, gradient method, Newton's method, convergence

DOI: 10.25728/arcRAS.2023.28.37.001

1. INTRODUCTION

Consider a linear discrete-time control system described by

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + Dw_k,\\
y_k &= Cx_k,\\
z_k &= C_1x_k,
\end{aligned}
\tag{1}
$$

with the following notations: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, and $C_1 \in \mathbb{R}^{r\times n}$ are given matrices of compatible dimensions; x_0 is an initial state; $x_k \in \mathbb{R}^n$ is the state vector; $y_k \in \mathbb{R}^l$ is the observed output; $z_k \in \mathbb{R}^r$ is the controlled output; $u_k \in \mathbb{R}^p$ is the control vector; $w_k \in \mathbb{R}^m$ is an exogenous disturbance bounded at each time instant:

$$
|w_k| \leqslant 1 \quad \text{for all } k = 0, 1, 2, \ldots. \tag{2}
$$

The problem of suppressing bounded exogenous disturbances is to find a stabilizing feedback control law that minimizes the value $\max_k |z_k|$. In this paper, we will design a linear static state- $u_k = Kx_k$ or output-feedback $u_k = Ky_k$ control law (if it exists).

The exact solution of this problem seems difficult; following the approach proposed in [1–3], we will find a suboptimal solution in terms of invariant ellipsoids. In this case, the original problem is treated as an optimization problem, where one variable is the gain matrix and the objective function to be minimized determines the performance criterion (the size of the ellipsoid containing the controlled output of the system). The corresponding approach goes back to the works [4, 5], devoted to linear quadratic control design.

This paper is a natural continuation of the publication [6], where the problem of suppressing bounded exogenous disturbances in a linear continuous-time control system was considered and solved from the same perspective.

The remainder of this paper is organized as follows. Section 2 discusses an algorithm for solving the analysis problem (finding the minimal bounding ellipsoid for the closed loop system). In Section 3, the control design problem is written as a nonconvex matrix optimization problem, and an iterative algorithm for solving it is formulated and justified. Section 4 provides an illustrative example.

2. ANALYSIS PROBLEM

Consider a discrete-time dynamic system described by

$$\begin{aligned} x_{k+1} &= Ax_k + Dw_k, \\ z_k &= Cx_k \end{aligned} \tag{3}$$

with a stable (Schur) matrix $A \in \mathbb{R}^{n\times n}$, an initial state x_0, the state vector $x_k \in \mathbb{R}^n$, the output $z_k \in \mathbb{R}^l$, and an exogenous disturbance $w_k \in \mathbb{R}^m$ that satisfies the constraint (2).

Recall that an ellipsoid of the form

$$\mathcal{E}_x = \left\{ x \in \mathbb{R}^n: \quad x^{\mathrm{T}} P^{-1} x \leqslant 1 \right\}, \quad P \succ 0,$$

is said to be invariant for system (3) if the condition $x_0 \in \mathcal{E}_x$ implies $x_k \in \mathcal{E}_x$ for all time instants $k = 1, 2, \ldots$. If $\mathcal{E}_x$ is an invariant ellipsoid with a matrix P, then the output z_k of system (3) with $x_0 \in \mathcal{E}_x$ belongs to the so-called bounding ellipsoid

$$\mathcal{E}_z = \left\{ z \in \mathbb{R}^r: \quad z^{\mathrm{T}} (CPC^{\mathrm{T}})^{-1} z \leqslant 1 \right\};$$

in the case $x_0 \notin \mathcal{E}_x$, the output will tend to this ellipsoid.

The analysis problem is to assess the effect of exogenous disturbances on the system output. Within the proposed approach, we are concerned with minimal ellipsoids containing the system output. A conventional minimality criterion for ellipsoids is the value $\operatorname{tr} CPC^{\mathrm{T}}$, equal to the sum of the squares of its semi-axes. The following result holds.

Theorem 1 [1, 3]. *Assume that the matrix A is Schur, $\rho = \max_i |\lambda_i(A)| < 1$, and the matrix $P(\alpha) \succ 0$, $\rho^2 < \alpha < 1$, satisfies the discrete Lyapunov equation*

$$\frac{1}{\alpha} APA^{\mathrm{T}} - P + \frac{1}{1-\alpha} DD^{\mathrm{T}} = 0.$$

Then the optimal bounding ellipsoid for system (3) *is obtained by minimizing the univariate function*

$$f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$$

on the interval $\rho^2 < \alpha < 1$; and if α^* is the minimum point and x_0 satisfies the condition $x_0^{\mathrm{T}} P^{-1}(\alpha^*) x_0 \leqslant 1$, then the estimate

$$|z_k| \leqslant \sqrt{f(\alpha^*)}, \quad k = 1, 2, \ldots,$$

holds.

The optimization problem formulated in Theorem 1 can be solved using Newton's method [7]. Let us choose an initial approximation $\rho^2(A) < \alpha_0 < 1$, e.g., $\alpha_0 = (1 + \rho^2(A))/2$, and apply the iterative process

$$\alpha_{j+1} = \alpha_j - \frac{f'(\alpha_j)}{f''(\alpha_j)}, \tag{4}$$

where

$$f'(\alpha) = \operatorname{tr} Y \left(\frac{1}{(1-\alpha)^2} DD^{\mathrm{T}} - \frac{1}{\alpha^2} APA^{\mathrm{T}} \right),$$

$$f''(\alpha) = 2\operatorname{tr} Y \left(\frac{1}{(1-\alpha)^3} DD^{\mathrm{T}} + \frac{1}{\alpha^3} A(P-X)A^{\mathrm{T}} \right),$$

and P, Y, and X are the solutions of the discrete Lyapunov equations

$$\frac{1}{\alpha} APA^{\mathrm{T}} - P + \frac{1}{1-\alpha} DD^{\mathrm{T}} = 0, \qquad \frac{1}{\alpha} A^{\mathrm{T}} YA - Y + C^{\mathrm{T}} C = 0,$$

and

$$\frac{1}{\alpha} AXA^{\mathrm{T}} - X + \frac{1}{(1-\alpha)^2} DD^{\mathrm{T}} - \frac{1}{\alpha^2} APA^{\mathrm{T}} = 0,$$

respectively.

The next theorem ensures the global convergence of this algorithm. It can be established by analogy with a similar result in [6].

Theorem 2. *In the method* (4),

$$|\alpha_j - \alpha^*| \leqslant \frac{f''(\alpha_0)}{2^j f''(\alpha^*)} |\alpha_0 - \alpha^*|, \qquad |\alpha_{j+1} - \alpha^*| \leqslant c|\alpha_j - \alpha^*|^2,$$

where $c > 0$ is some constant.

3. DESIGN PROBLEM

Returning to system (1), we suppose that the matrices D and C_1 are square and nonsingular.[1] The problem is to find a linear static output-feedback control law

$$u_k = Ky_k$$

(in the case $C = I$, a linear static state-feedback control law) that stabilizes the closed loop system (1) and suppresses the exogenous disturbances (2) by minimizing the bounding ellipsoid for the controlled output z_k. As an optimality criterion we choose the value

$$\operatorname{tr} C_1 P C_1^{\mathrm{T}} + \rho \|K\|_F^2,$$

where the first component describes the size of the bounding ellipsoid and the second one is a control penalty to avoid large values of the gain matrix. (The coefficient $\rho > 0$ adjusts its significance.)

Due to Theorem 1, the original problem is reduced to the matrix optimization problem

$$\min f(K, \alpha), \quad f(K, \alpha) = \operatorname{tr} C_1 P C_1^{\mathrm{T}} + \rho \|K\|_F^2$$

subject to the constraint

$$\frac{1}{\alpha}(A + BKC)P(A + BKC)^{\mathrm{T}} - P + \frac{1}{1-\alpha} DD^{\mathrm{T}} = 0 \tag{5}$$

with respect to the matrix variables $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{p\times n}$ and the scalar parameter $0 < \alpha < 1$.

According to Section 2, minimization with respect to the parameter α can be performed rather effectively. (It suffices to replace the matrix A by $A + BKC$.) Therefore, we will focus on minimizing the function

$$f(K) = \min_{\alpha} f(K, \alpha).$$

[1] No doubt, this technical assumption can be relaxed; for the time being, the objective is to establish simple and visual results.

Assumption. Let K_0 be a known stabilizing controller, i.e., the matrix $A + BK_0C$ is Schur.

Note that the function $f(K)$ is well-defined and positive on the set $\mathcal{S}$ of stabilizing controllers. Its definitional domain $\mathcal{S}$ can be nonconvex and disconnected whereas its boundaries can be nonsmooth. Here, the situation completely matches the continuous-time case; see [6].

Lemma 1. *The function $f(K)$ is coercive on the set $\mathcal{S}$ of stabilizing controllers (i.e., it tends to infinity on its boundary) and, moreover,*

$$f(K) \geqslant \frac{1}{1-\rho^2(A+BKC)} \frac{\lambda_{\min}(CC^{\mathrm{T}})}{1-\sigma^2_{\min}(A+BKC)} \|D\|_F^2, \tag{6}$$
$$f(K) \geqslant \rho\|K\|^2.$$

Corollary 1. *The level set*

$$\mathcal{S}_0 = \{K \in \mathcal{S}\colon \quad f(K) \leqslant f(K_0)\}$$

is bounded for any controller $K_0 \in \mathcal{S}$.

On the other hand, the function $f(K)$ has a minimum point on the set $\mathcal{S}_0$ (as a continuous function on a compact set), but the set $\mathcal{S}_0$ shares no points with the boundary of $\mathcal{S}$ due to (6). It will be demonstrated below that $f(K)$ is differentiable on $\mathcal{S}_0$; hence, the following result is valid.

Corollary 2. *There exists a minimum point K_* on the set S, and the gradient vanishes at this point.*

The gradient and Hessian of the function $f(K, \alpha)$ have properties described by the two lemmas below.

Lemma 2. *The function $f(K, \alpha)$ is well-defined and differentiable on the set $\mathcal{S}$ of stabilizing controllers K for $\rho^2(A + BKC) < \alpha < 1$. In addition,*

$$\frac{1}{2}\nabla_K f(K,\alpha) = \rho K + \frac{1}{\alpha} B^{\mathrm{T}} Y (A+BKC) P C^{\mathrm{T}}, \tag{7}$$
$$\nabla_\alpha f(K,\alpha) = \operatorname{tr} Y \left(\frac{1}{(1-\alpha)^2} DD^{\mathrm{T}} - \frac{1}{\alpha^2}(A+BKC)P(A+BKC)^{\mathrm{T}} \right),$$

where the matrices P and Y are the solutions of the discrete Lyapunov equations

$$\frac{1}{\alpha}(A+BKC)P(A+BKC)^{\mathrm{T}} - P + \frac{1}{1-\alpha} DD^{\mathrm{T}} = 0 \tag{8}$$

and

$$\frac{1}{\alpha}(A+BKC)^{\mathrm{T}} Y (A+BKC) - Y + C_1^{\mathrm{T}} C_1 = 0, \tag{9}$$

respectively.

The function $f(K, \alpha)$ achieves minimum at an inner point of the set $\mathcal{S} \times (\rho^2(A + BKC), 1)$. This point is given by the conditions

$$\nabla_K f(K,\alpha) = \nabla_\alpha f(K,\alpha) = 0.$$

In addition, $f(K, \alpha)$ as a function of α is strictly convex on $\rho^2(A + BKC) < \alpha < 1$ and achieves minimum at an inner point of this interval.

Lemma 3. *The function $f(K,\alpha)$ is twice differentiable with respect to K, and the action of its Hessian on an arbitrary matrix*[2] *$E \in \mathbb{R}^{p\times l}$ is given by*

$$\frac{1}{2}\nabla_K^2 f(K,\alpha)[E,E] = \rho\langle E,E\rangle + \frac{1}{\alpha}\langle B^{\mathrm{T}}YBECPC^{\mathrm{T}},E\rangle + \frac{2}{\alpha}\langle B^{\mathrm{T}}Y(A+BKC)P'C^{\mathrm{T}},E\rangle,$$

where P' is the solution of the discrete Lyapunov equation

$$\frac{1}{\alpha}(A+BKC)P'(A+BKC)^{\mathrm{T}} - P' + \frac{1}{\alpha}\Big((A+BKC)P(BEC)^{\mathrm{T}} + BECP(A+BKC)^{\mathrm{T}}\Big) = 0. \quad (10)$$

The gradient of $f(K,\alpha)$ as a function of K is not Lipschitz on the set $\mathcal{S}$ of stabilizing controllers. However, like in the continuous-time case, it has the Lipschitz property on the subset $\mathcal{S}_0$. (This result can be easily obtained.)

The above properties of the objective function allow constructing a minimization method and justifying its convergence. That is, we propose an iterative approach to solve the problem that involves the gradient method with respect to the variable K and Newton's method with respect to the variable α.

The algorithm includes several steps as follows.

1. Choose some values of the parameters $\varepsilon > 0$, $\gamma > 0$, $0 < \tau < 1$, and the initial stabilizing approximation K_0. Calculate
$$\alpha_0 = \frac{1+\rho^2(A+BK_0C)}{2}.$$
2. On the jth iteration, the controller K_j and the value α_j are given. Calculate the matrix $A_j = A + BK_jC$, solve equations (8) and (9) to find the matrices P and Y. Calculate the gradient
$$H_j = \nabla_K f(K_j,\alpha_j)$$
from the relation (7).
If $\|H_j\| \leqslant \varepsilon$, then take the controller K_j as the approximate solution.
3. Perform the gradient method step:
$$K_{j+1} = K_j - \gamma_j H_j.$$
Adjust the step length $\gamma_j > 0$ by fractionating γ until the following conditions are satisfied:
a. K_{j+1} is a stabilizing controller, i.e., the matrix $(A+BK_{j+1}C)/\sqrt{\alpha_j}$ is Schur.
b. $f(K_{j+1}) \leqslant f(K_j) - \tau\gamma_j\|H_j\|^2$.
4. Minimize $f(K_{j+1},\alpha)$ with respect to α and find α_{j+1}. Revert to Step 2.

This algorithm converges in the following sense.

Theorem 3. *Only a finite number of fractions are realized for γ_j at each iteration of the algorithm, the function $f(K_j)$ is monotonically decreasing, and its gradient vanishes with an exponential rate (like a geometric progression):*

$$\lim_{j\to\infty}\|H_j\| = 0.$$

The proof is completely analogous to the continuous-time case and uses the common gradient method analysis scheme for the unconstrained minimization of functions with a Lipschitz gradient [8].

[2] In the sense of the second derivative in a direction.

4. EXAMPLE

Consider a system of the form (1) with the matrices

$$A = \begin{pmatrix} 0.9950 & 0.0050 & 0.0998 & 0.0002 \\ 0.0050 & 0.9950 & 0.0002 & 0.0998 \\ -0.0997 & 0.0997 & 0.9950 & 0.0050 \\ 0.0997 & -0.0997 & 0.0050 & 0.9950 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.0050 \\ 0.0000 \\ 0.0998 \\ 0.0002 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.0050 & 0.0000 \\ 0.0000 & 0.0050 \\ 0.0998 & 0.0002 \\ 0.0002 & 0.0998 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is a slight modification of Example 4.3.2 from the monograph [3].

Let $\rho = 0.1$ and choose

$$K_0 = \begin{pmatrix} -2.9823 \\ -3.9608 \end{pmatrix}$$

as an initial stabilizing controller.

The iterative process terminated in the 25th iteration and yielded the controller

$$K_* = \begin{pmatrix} -0.6519 \\ -1.8166 \end{pmatrix}$$

and the corresponding bounding ellipse for the controlled output of the system with the matrix

$$\begin{pmatrix} 19.2309 & -3.4643 \\ -3.4643 & 10.3506 \end{pmatrix}.$$

The dynamics of the iterative process are shown in Fig. 1.

For the initial stabilizing controller

$$K_0' = \begin{pmatrix} -0.3675 \\ -0.7106 \end{pmatrix},$$

in the 24th iteration we obtain the controller

$$K_*' = \begin{pmatrix} -0.6527 \\ -1.8166 \end{pmatrix}$$

and the corresponding bounding ellipse with the matrix

$$\begin{pmatrix} 19.2293 & -3.4638 \\ -3.4638 & 10.3543 \end{pmatrix}.$$

Note that the controllers K_* and K_*' differ in norm by fractions of a percent. The same applies to the bounding ellipses when contrasted by the trace criterion.

For comparison, we solve the same problem by constructing a dynamic feedback controller

$$u_k = K\widehat{x}_k$$

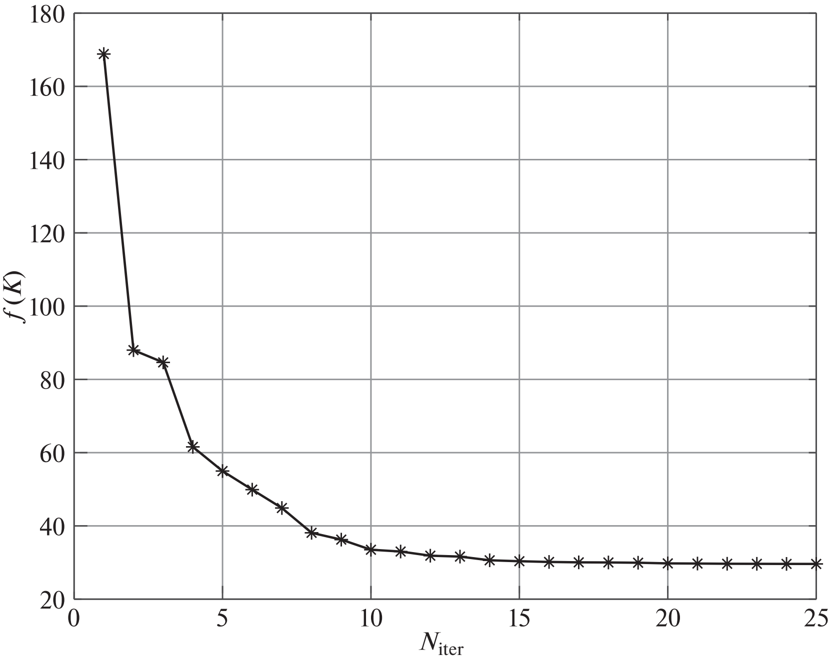

Fig. 1. Optimization procedure.

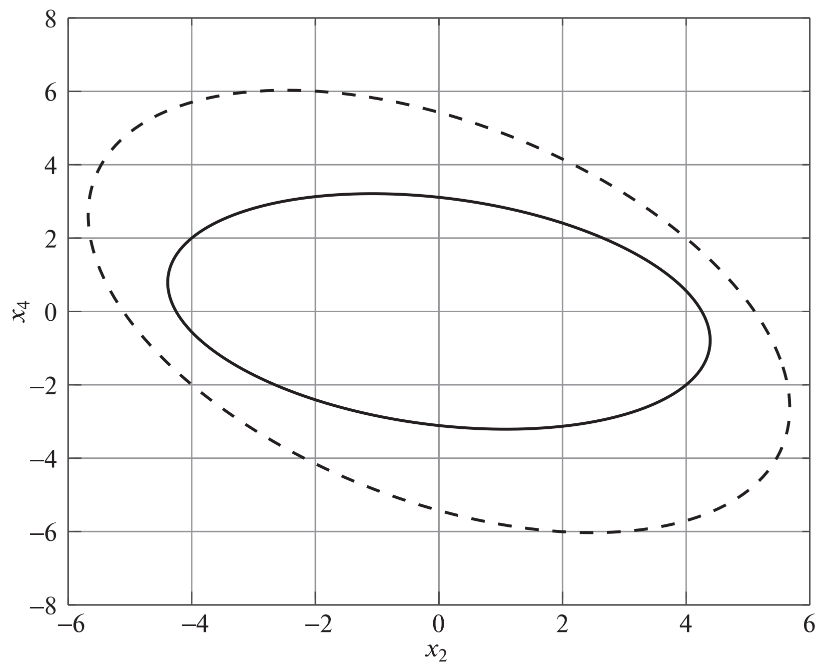

Fig. 2. Bounding ellipses.

using the observer

$$\widehat{x}_{k+1} = A\widehat{x}_k + Bu_k + L(y_k - C\widehat{x}_k), \quad \widehat{x}_0 = 0.$$

Following the approach [3] and the technique of linear matrix inequalities (LMIs), we calculate the gain matrix

$$K = \begin{pmatrix} -39.0055 & -46.7193 & -8.5074 & -98.0176 \end{pmatrix},$$

the observer matrix

$$L = \begin{pmatrix} 0.5655 & 0.0759 \\ -6.7183 & 1.8722 \\ -2.2061 & 1.0573 \\ -2.8715 & 0.7224 \end{pmatrix},$$

and the matrix

$$\begin{pmatrix} 32.2165 & -14.9238 \\ -14.9238 & 36.3654 \end{pmatrix}$$

of the ellipse containing the controlled output.

These problem statements have a small technical difference: in the latter case, the regularizing term $\rho\|K\|_F^2$ is eliminated from the objective function and an additional term with control is introduced into the regulated output of the system for the same purpose: $z_k = C_1x_k + B_1u_k$.

In Fig. 2, the solid line shows the bounding ellipse yielded by the iterative procedure whereas the dotted line the one provided by the dynamic controller. The rather large difference in the sizes of the ellipses can be explained as follows: when constructing a dynamic feedback controller, it is necessary to roughen several things in order to linearize the matrix inequalities, which leads to excessive conservatism.

5. CONCLUSIONS

This paper has proposed a new controller design approach for the optimal suppression of bounded exogenous disturbances in a linear discrete-time system. It is based on reducing the original problem to a matrix optimization problem with the gain matrix as one variable. Next, this problem is solved using the gradient method. Its convergence has been theoretically justified for several important special cases. A numerical example has been presented to demonstrate the effectiveness of the proposed procedure.

The problem of suppressing exogenous disturbances has been considered under fairly strict restrictions. In particular, it has been assumed that the dimension of disturbances and controlled outputs coincides with the number of states. However, the method quite effectively works in the absence of such restrictions. An important task is to justify the method in this case as well.

Since the definitional domain of the function $f(K)$ may even be disconnected, it is difficult to expect convergence to a global minimum. However, for the problem with state-feedback control, as in the continuous case, one can apparently expect that the objective function satisfies the gradient dominance condition and, hence, global convergence to a unique minimum point.

APPENDIX

Proof of Lemma 1. Consider a sequence of stabilizing controllers $\{K_j\} \in \mathcal{S}$ such that $K_j \to K \in \partial\mathcal{S}$, i.e., $\rho(A + BKC) = 1$. In other words, for any $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$|\rho(A + BK_jC) - \rho(A + BKC)| = 1 - \rho(A - BK_jC) < \varepsilon$$

for all $j \geqslant N(\varepsilon)$.

Let P_j be the solution of equation (5) associated with the controller K_j:

$$\frac{1}{\alpha_j}(A + BK_jC)P_j(A + BK_jC)^{\mathrm{T}} - P_j + \frac{1}{1-\alpha_j}DD^{\mathrm{T}} = 0.$$

Also, let Y_j be the solution of the dual discrete Lyapunov equation

$$\frac{1}{\alpha_j}(A+BK_jC)^{\mathrm{T}}Y_j(A+BK_jC)-Y_j+C_1C_1^{\mathrm{T}}=0.$$

Using [6, Lemmas A.1 and A.2] and [7, Lemma A.1.2], we have

$$f(L_j)=\operatorname{tr}C_1P_jC_1^{\mathrm{T}}+\rho\|K_j\|_F^2\geqslant\operatorname{tr}P_jC_1C_1^{\mathrm{T}}=\operatorname{tr}\left(Y_j\frac{1}{1-\alpha_j}DD^{\mathrm{T}}\right)$$

$$\geqslant\frac{1}{1-\alpha_j}\lambda_{\min}(Y_j)\|D\|_F^2\geqslant\frac{1}{1-\alpha_j}\frac{\lambda_{\min}(C_1C_1^{\mathrm{T}})}{1-\sigma_{\min}^2(A+BK_jC)}\|D\|_F^2$$

$$\geqslant\frac{1}{1-\rho^2(A+BK_jC)}\frac{\lambda_{\min}(C_1C_1^{\mathrm{T}})}{1-\sigma_{\min}^2(A+BK_jC)}\|D\|_F^2$$

$$\geqslant\frac{1}{\varepsilon}\frac{1}{1+\rho(A+BK_jC)}\frac{\lambda_{\min}(C_1C_1^{\mathrm{T}})}{1-\sigma_{\min}^2(A+BK_jC)}\|D\|_F^2\xrightarrow[\varepsilon\to 0]{}+\infty$$

since $\rho^2(A+BK_jC)<\alpha_j<1$.

On the other hand,

$$f(K_j)=\operatorname{tr}C_1P_jC_1^{\mathrm{T}}+\rho\|K_j\|_F^2\geqslant\rho\|K_j\|_F^2\geqslant\rho\|K_j\|^2\xrightarrow[\|K_j\|\to+\infty]{}+\infty.$$

The proof of Lemma 1 is complete.

Proof of Lemma 2. Differentiation with respect to α is performed in accordance with the results of Section 2, with A replaced by $A+BKC$.

We add the increment ΔK for K in equation (5) and denote the corresponding increment of P by ΔP:

$$\frac{1}{\alpha}(A+B(K+\Delta K)C)(P+\Delta P)(A+B(K+\Delta K)C)^{\mathrm{T}}-(P+\Delta P)+\frac{1}{1-\alpha}DD^{\mathrm{T}}=0.$$

Leaving the notation ΔP for the principal part of the increment, we have

$$\frac{1}{\alpha}\Big((A+BKC)P(A+BKC)^{\mathrm{T}}+B\Delta KCP(A+BKC)^{\mathrm{T}}$$
$$+(A+BKC)P(B\Delta KC)^{\mathrm{T}}+(A+BKC)\Delta P(A+BKC)^{\mathrm{T}}\Big)$$
$$-(P+\Delta P)+\frac{1}{1-\alpha}DD^{\mathrm{T}}=0.$$

Subtracting equation (5) from this equation gives

$$\frac{1}{\alpha}(A+BKC)\Delta P(A+BKC)^{\mathrm{T}}-\Delta P$$
$$+\frac{1}{\alpha}\Big((A+BKC)P(B\Delta KC)^{\mathrm{T}}+B\Delta KCP(A+BKC)^{\mathrm{T}}\Big)=0. \quad \text{(A.1)}$$

The increment of $f(K)$ is calculated by linearizing the corresponding terms:

$$\Delta f(K)=f(K)-f(K+\Delta K)$$
$$=\operatorname{tr}C_1(P+\Delta P)C_1^{\mathrm{T}}+\rho\|K+\Delta K\|_F^2-(\operatorname{tr}C_1PC_1^{\mathrm{T}}+\rho\|K\|_F^2)$$
$$=\operatorname{tr}C_1\Delta PC_1^{\mathrm{T}}+\rho\operatorname{tr}K^{\mathrm{T}}\Delta K+\rho\operatorname{tr}(\Delta K)^{\mathrm{T}}K=\operatorname{tr}\Delta PC_1^{\mathrm{T}}C_1+2\rho\operatorname{tr}K^{\mathrm{T}}\Delta K.$$

Due to [6, Lemma A.1], from the dual equations (A.1) and (9) it follows that

$$\begin{aligned}\Delta f(K) &= 2\mathrm{tr}\, Y\frac{1}{\alpha}B\Delta KCP(A+BKC)^{\mathrm{T}} + 2\rho\mathrm{tr}\, K^{\mathrm{T}}\Delta K\\ &= 2\mathrm{tr}\left(\rho K^{\mathrm{T}} + \frac{1}{\alpha}CP(A+BKC)^{\mathrm{T}}YB\right)\Delta K\\ &= 2\left\langle \rho K + \frac{1}{\alpha}B^{\mathrm{T}}Y(A+BKC)PC^{\mathrm{T}}, \Delta K\right\rangle.\end{aligned}$$

Thus, we arrive at (7). The proof of Lemma 2 is complete.

Proof of Lemma 3. The value

$$\nabla_K^2 f(K,\alpha)[E,E] = \langle \nabla_K^2 f(K,\alpha)[E], E\rangle,$$

is calculated by differentiating $\nabla_K f(K,\alpha)[E] = \langle \nabla_K f(K,\alpha), E\rangle$ in the direction $E \in \mathbb{R}^{p\times l}$.

For this purpose, linearizing the corresponding terms, we calculate the increment of $\nabla_K f(K,\alpha)[E]$ in the direction E:

$$\begin{aligned}&\Delta\nabla_K f(K,\alpha)[E]\\ &= 2\left(\rho(K+\delta E) + \frac{1}{\alpha}B^{\mathrm{T}}(Y+\Delta Y)(A+B(K+\delta E)C)(P+\Delta P)C^{\mathrm{T}}\right)\\ &\quad -2\left(\rho K + \frac{1}{\alpha}B^{\mathrm{T}}Y(A+BKC)PC^{\mathrm{T}}\right)\\ &= 2\delta\Big(\rho E + \frac{1}{\alpha}B^{\mathrm{T}}(YBECP + Y'(K)[E](A+BKC)P\\ &\qquad + Y(A+BKC)P'(K)[E])C^{\mathrm{T}}\Big),\end{aligned}$$

where

$$\begin{aligned}\Delta P &= P(K+\delta E) - P(K) = \delta P'(K)[E],\\ \Delta Y &= Y(K+\delta E) - Y(K) = \delta Y'(K)[E].\end{aligned}$$

Thus, with $P' = P'(K)[E]$ and $Y' = Y'(K)[E]$, we have

$$\begin{aligned}&\frac{1}{2}\nabla_K^2 f(K,\alpha)[E,E]\\ &= \left\langle \rho E + \frac{1}{\alpha}B^{\mathrm{T}}(YBECP + Y'(A+BKC)P + Y(A+BKC)P')C^{\mathrm{T}}, E\right\rangle.\end{aligned}$$

Furthermore, $P = P(K)$ is the solution of the discrete Lyapunov equation (5). We write it in increments in the direction E:

$$\frac{1}{\alpha}(A+B(K+\delta E)C)(P+\delta P')(A+B(K+\delta E)C)^{\mathrm{T}} - (P+\delta P') + \frac{1}{1-\alpha}DD^{\mathrm{T}} = 0$$

or

$$\begin{aligned}&\frac{1}{\alpha}\Big((A+BKC)P(A+BKC)^{\mathrm{T}} + (A+BKC)\delta P'(A+BKC)^{\mathrm{T}}\\ &\qquad + (A+BKC)P(B\delta EC)^{\mathrm{T}} + B\delta ECP(A+BKC)^{\mathrm{T}}\Big)\\ &\qquad - (P+\delta P') + \frac{1}{1-\alpha}DD^{\mathrm{T}} = 0.\end{aligned}$$

In view of (5), this expression yields equation (10).

Similarly, $Y = Y(K)$ is the solution of the discrete Lyapunov equation (9). We write it in increments in the direction E:

$$\frac{1}{\alpha}(A + B(K + \delta E)C)^{\mathrm{T}}(Y + \delta Y')(A + B(K + \delta E)C) - (Y + \delta Y') + C_1^{\mathrm{T}} C_1 = 0$$

or

$$\frac{1}{\alpha}\Big((A + BKC)^{\mathrm{T}} Y (A + BKC) + (A + BKC)^{\mathrm{T}} \delta Y'(A + BKC) \\ + (A + BKC)^{\mathrm{T}} Y B\delta EC + (B\delta EC)^{\mathrm{T}} Y (A + BKC)\Big) - (Y + \delta Y') + C_1^{\mathrm{T}} C_1 = 0.$$

Due to (9), we obtain

$$\frac{1}{\alpha}(A + BKC)^{\mathrm{T}} Y'(A + BKC) - Y' \\ + \frac{1}{\alpha}\Big((A + BKC)^{\mathrm{T}} Y BEC + (BEC)^{\mathrm{T}} Y (A + BKC)\Big) = 0. \tag{A.2}$$

From (10) and (A.2) it follows that

$$\operatorname{tr} P'(A + BKC)^{\mathrm{T}} Y BEC = \operatorname{tr} Y' BECP(A + BKC)^{\mathrm{T}},$$

so

$$\frac{1}{2}\nabla_K^2 f(K, \alpha)[E, E] = \rho\langle E, E\rangle + \frac{1}{\alpha}\langle B^{\mathrm{T}} Y BECPC^{\mathrm{T}}, E\rangle + \frac{2}{\alpha}\langle B^{\mathrm{T}} Y (A + BKC)P'C^{\mathrm{T}}, E\rangle.$$

The proof of Lemma 3 is complete.

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This paper was recommended for publication by V.M. Glumov, a member of the Editorial Board